MATH2050C Assignment 4

Deadline: Feb 5, 2024.

Hand in: 3.2 no. 14b, 16d; 3.3 no. 5, 12c; Suppl Problems no 1, 2, 3.

Section 3.2 no. 14ab, 16bd, 19bd;

Section 3.3 no. 3, 5, 7, 10, 12ac.

Supplementary Problems

1. Suppose that $\lim_{n\to\infty} x_n = x$. Prove that
$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x\ .$$

2. Determine the limit of
$$\left(1 - \frac{a}{n^2}\right)^n\ ,\ a > 0\ .$$
Hint: Use Bernoulli's inequality.

3. Show the limit of $(1 - a/n)^n,\ a > 0$ is equal to $1/E(a)$. Hint: Use (2).

4. Prove that e is irrational. Hint: Use the inequality $0 < e-(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}) < \frac{1}{k\times k!}$.

See next page for more.

The Exponential

First we note the following more detailed form of Theorem 3.2.11.

Theorem 3.2.11' let $\{x_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} x_{n+1}/x_n = L \in [0,1)$. Then $\lim_{n\to\infty} x_n = 0$ and there is some M such that

$$x_1 + x_2 + \cdots + x_n \le M, \quad \forall n \ .$$

Proof Fix $\gamma, 0 \le L < \gamma < 1$. For $\varepsilon = \gamma - L > 0$, there is some n_0 such that $x_{n+1}/x_n \le \gamma$ for all $n \ge n_0$. Therefore,

$$0 < x_{n+1} \le \gamma x_n \le \gamma^2 x_{n-1} \le \cdots \le \gamma^{n-n_0+1} x_{n_0} \ ,$$

which implies $x_n \to 0$ as $n \to \infty$ by Squeeze Theorem. Furthermore,

$$x_{n_0} + x_{n_0+1} \cdots + x_n \le x_{n_0} + \gamma x_{n_0} + \cdots + \gamma^{n-n_0} x_{n_0} \le \frac{x_{n_0}}{1-\gamma} \equiv M \ .$$

Theorem For $a > 0$, the sequence $x_n = (1 + a/n)^n$ is strictly increasing and bounded from above. Consequently,

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$$

exists.

Proof By binomial theorem,

$$\begin{aligned} x_n &= 1 + n\frac{a}{n} + \frac{n(n-1)}{2!}\frac{a^2}{n^2} + \cdots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\frac{a^k}{k!} + \cdots + \frac{a^n}{n^n} \\ &= 1 + a + \frac{1}{2!}\left(1 - \frac{1}{n}\right)a^2 + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)a^k + \\ &\quad \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)a^n \ . \end{aligned}$$

x_{n+1} is obtained by replacing the n's in the formula above by $n+1$. By a term by term comparison, we see that $x_n < x_{n+1}$, that is, $\{x_n\}$ is strictly increasing. Next, from this formula we also have

$$x_n < 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!} \ .$$

By Theorem 3.2.11' above, $a^{n+1}/(n+1)! \times n!/a^n = a/(n+1) \to 0$ as $n \to \infty$, we conclude that there is some M such that $1 + a + \dfrac{a^2}{2!} + \cdots + \dfrac{a^n}{n!} \le M$ for all n. By Monotone Convergence Theorem the limit of x_n exists and is equal to its supremum.

For $a \ge 0$, we define a function $E(a)$ by setting

$$E(a) = \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n \ .$$

We also write $e = E(1)$ and call it the exponential. e and π are two of the most important constants in science.

Theorem For each k,

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right) \le \frac{1}{k \times k!} \ .$$

Proof From the proof above, for $k < n$,

$$\begin{aligned}
0 &< \left(1+\frac{a}{n}\right)^n - \left(1+a+\frac{1}{2!}a^2+\frac{1}{3!}a^3\cdots+\frac{1}{k!}a^k\right)\\
&< \frac{a^k}{k!}\left(\frac{a}{k+1}+\frac{a^2}{(k+2)(k+1)}+\frac{a^3}{(k+3)(k+2)(k+1)}+\cdots\right)\\
&< \frac{a^k}{k!}\left(\frac{a}{k+1}+\frac{a^2}{(k+1)^2}+\frac{a^3}{(k+1)^3}+\cdots\right)\\
&= \frac{a^k}{k!}\frac{a}{k+1}\frac{1}{1-a/(k+1)}\\
&= \frac{a^{k+1}}{k!}\frac{1}{k+1-a}\ ,
\end{aligned}$$

and the result follows by taking $a = 1$.